Journal of Complexity 26 (2010) 316–333

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Improved generalized differentiability conditions for Newton-like methods

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ARTICLE INFO

Article history:
Received 5 September 2009
Accepted 15 December 2009
Available online 4 January 2010

Keywords:
Newton-like method
Majorizing sequence
Semilocal convergence
Chandrasekhar nonlinear integral equation
Radiative transfer
Differential equation with Green's kernel

ABSTRACT

We provide a semilocal convergence analysis for Newton-like methods using the ω-versions of the famous Newton–Kantorovich theorem (Argyros (2004) [1], Argyros (2007) [3], Kantorovich and Akilov (1982) [13]). In the special case of Newton's method, our results have the following advantages over the corresponding ones (Ezquerro and Hernaández (2002) [10], Proinov (2010) [17]) under the same information and computational cost: finer error estimates on the distances involved; at least as precise information on the location of the solution, and weaker sufficient convergence conditions.

Numerical examples, involving a Chandrasekhar-type nonlinear integral equation as well as a differential equation with Green's kernel are provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a subset $\mathcal{D}$ of a Banach space $\mathcal{X}$, with values in a Banach space $\mathcal{Y}$, and $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a Fréchet-differentiable operator.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by

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0885-064X/$ – see front matter © 2010 Published by Elsevier Inc.
doi:10.1016/j.jco.2009.12.001

difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use the Newton-like method:

$$x_{n+1} = x_n - A(x_n)^{-1} F(x_n), \quad (n \geq 0), \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $(x \in \mathcal{D})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. $A(x)$ is an approximation to the Fréchet-derivative $F'(x)$ of operator F [3].

If we set

$$A(x) = F'(x), \quad (x \in \mathcal{D}), \tag{1.3}$$

we obtain the Newton–Kantorovich method;

$$A(x_n) = [x_{n-1}, x_n; F], \tag{1.4}$$

we obtain the Secant method;

$$\begin{aligned} &A(x) = [x, g(x); F], \\ &g : \mathcal{X} \longrightarrow \mathcal{X} \quad \text{is a continuous operator,} \end{aligned} \tag{1.5}$$

we obtain Steffensen's method. Other choices of operator A can be found in [1–6].

A current survey on local as well as semilocal convergence theorems for Newton-like methods (1.2) under various Lipschitz-type assumptions can be found in [3,7–12], and the references therein (see also [14–20]).

Here, in particular we are motivated by optimization considerations, and the elegant works by Ezquerro, Hernández in [10], and Proinov in [17]. They proved semilocal convergence results for the special case $A(x) = F'(x)$, $(x \in \mathcal{D})$ by using the affine invariant condition

$$\| F'(x_0)^{-1} (F'(x) - F'(y)) \| \leq \omega(\| x - y \|), \quad \text{for all } x, y \in \mathcal{D} \tag{1.6}$$

where, ω is a nondecreasing, non-negative function on $[0, \infty)$.

Moreover, they considered a function h on $[0, 1]$ such that:

$$\omega(s\,t) \leq h(s)\omega(t) \quad \text{for all } s \in [0, 1], \text{ and } t \in [0, +\infty). \tag{1.7}$$

This condition has been successfully used to sharpen the error bounds obtained for particular expressions [10] (see also [17, Section 7]). Note that such a function h always exists. Indeed, if ω is a nonzero function on $\mathcal{J} = [0, +\infty)$, then one can define function $h : [0, 1] \longrightarrow \mathbb{R}$ by

$$h(s) = \sup \left\{ \frac{\omega(s\,t)}{\omega(t)} : t \in [0, \infty), \text{with } \omega(t) > 0 \right\}. \tag{1.8}$$

Clearly, function h so defined satisfies (1.7), and has the following properties [17]:

- $h(0) = 0$, $h(1) = 1$ provided that $\omega(0) = 0$;
- h is nondecreasing on $[0, 1]$ provided that ω is nondecreasing on $\mathcal{J}$;
- h is continuous on $[0, 1]$ provided that ω is nondecreasing on $\mathcal{J}$;
- h is identical to 1 on $[0, 1]$ if ω is nondecreasing on $\mathcal{J}$ and $\omega(0) > 0$.

Several choices of function h can be found in [17].

The study is organized as follows: In Section 2, we provide a semilocal convergence theorem for the Newton-like method (1.2), whereas in Section 3, we provide an extension of this result to solve more general equations than (1.1). Finally, in Section 4, we consider special cases and applications. In particular, in order for us to compare our results with the corresponding ones in [10,17], we set $A(x) = F'(x)$, $(x \in \mathcal{D})$ to show that our results have the following advantages under the same information, and computational cost:

- Finer error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$, $(n \geq 0)$;
- At least as precise information on the location of the solution;
- Weaker sufficient conditions convergence.

Numerical examples are also provided, involving Chandrasekhar-type nonlinear integral equations as well as differential equations involving Green's function, where our results apply, but earlier ones do not [10,17].

2. Semilocal convergence analysis of the Newton-like method

We provide the main semilocal convergence result for the Newton-like method (1.2).

Theorem 2.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator, and let $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an approximation of $F'(x)$. Assume that there exist an open convex subset $\mathcal{D}_0$ of $\mathcal{D}$, a vector $x_0 \in \mathcal{D}_0$, a bounded inverse Γ of A $(=A(x_0))$, continuous, nondecreasing, non-negative functions ω, ω_0, ω_1, ω_2 on $[0, +\infty)$, h, h_0, h_1 on $[0, 1]$, and non-negative constants η, ℓ_0, ℓ_1, ℓ_2, such that, for all $x, y \in \mathcal{D}_0$, $t \in [0, 1]$, $s \in [0, \infty)$, the following conditions hold:*

$$\|\Gamma F(x_0)\| \leq \eta, \tag{2.1}$$

$$\|\Gamma[F'(x) - F'(y)]\| \leq \omega(\|x - y\|), \tag{2.2}$$

$$\omega(t\, s) \leq h(t)\omega(s), \tag{2.3}$$

$$\|\Gamma[A(x) - A(x_0)]\| \leq \omega_0(\|x - x_0\|) + \ell_0, \tag{2.4}$$

$$\omega_0(t\, s) \leq h_0(t)\omega_0(s), \tag{2.5}$$

$$\|\Gamma[F'(x) - F(x_0)]\| \leq \omega_1(\|x - x_0\|) + \ell_1, \tag{2.6}$$

$$\omega_1(t\, s) \leq h_1(t)\omega_1(s), \tag{2.7}$$

$$\|\Gamma[F'(x) - A(x)]\| \leq \omega_2(\|x - x_0\|) + \ell_2, \tag{2.8}$$

$$\ell_0 + \ell_2 < 1, \tag{2.9}$$

$$\ell_1 < 1. \tag{2.10}$$

Set:

$$H = \int_0^1 h(t)\mathrm{d}t, \qquad H_1 = \int_0^1 h_1(t)\mathrm{d}t$$

$$c_0 = \frac{H_1\omega_1(\eta) + \omega_2(0) + \ell_2}{1 - \ell_0 - \omega_0(\eta)}, \qquad c = c(r) = \frac{H\,\omega(\eta) + \omega_2(r) + \ell_2}{1 - \ell_0 - \omega_0(r)}.$$

We also assume that the scalar equation

$$\left(1 + c_0(1 + c) + \frac{c^3}{1 - c}\right)\eta = r \tag{2.11}$$

has a minimum zero r_0, such that:

$$r_0 > \eta, \tag{2.12}$$

$$c(r_0) < 1, \tag{2.13}$$

$$\omega_0(r_0) < 1 - \ell_0, \tag{2.14}$$

and

$$\overline{U}(x_0, r_0) = \{x \in \mathcal{X} : \|x - x_0\| \le r_0\} \subseteq \mathcal{D}_0. \tag{2.15}$$

Then, sequences $\{x_n\}$ $(n \ge 0)$ generated by the Newton-like method (1.2) is well defined, remains in $\overline{U}(x_0, r_0)$ for all $n \ge 0$, and converges to a solution $x^\star \in \overline{U}(x_0, r_0)$ of equation $F(x) = 0$.

Moreover, the following estimates hold for all $n \ge 0$:

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \tag{2.16}$$

and

$$\|x_n - x^\star\| \le t^\star - t_n, \tag{2.17}$$

where, scalar sequence $\{t_n\}$, $(n \ge 0)$, and $t^\star$ are given by:

$$\begin{aligned} &t_0 = 0, \qquad t_1 = \eta \qquad t_2 = t_1 + c_0\,\eta \\ &t_{n+1} = t_n + \frac{H\,\omega(t_n - t_{n-1}) + \omega_2(t_{n-1}) + \ell_2}{1 - \ell_0 - \omega_0(t_n)}(t_n - t_{n-1}), \quad (n \ge 2), \end{aligned} \tag{2.18}$$

and

$$t^\star = \lim_{n \to \infty} t_n \le r_0. \tag{2.19}$$

Furthermore, the solution $x^\star$ of Eq. (1.1) is unique in

$$\mathcal{D}_1 = U(x_0, r_1) \cap \mathcal{D}_0 \tag{2.20}$$

where, r_1 is the positive root of equation

$$2\,\omega_1(r_0 + r_1) + \int_{1/2}^{1} h_1(t)\mathrm{d}t + \ell_1 = 1. \tag{2.21}$$

Proof. By hypotheses (2.11)–(2.14), and definition (2.18), we have $t_0 \le t_1 \le t_2 \le r_0$.

Let us assume $t_{k-1} \le t_k \le r_0$, for all $k \le \eta$. Then, by the definition of ω, h functions, (2.18), and the induction hypotheses, we obtain $t_k \le t_{k+1}$. We also have:

$$\begin{aligned} t_{k+1} &\le t_k + c(t_k - t_{k-1}) \\ &\le t_{k-1} + c(t_{k-1} - t_{k-2}) + c(t_k - t_{k-1}) \\ &\le t_2 + c(t_2 - t_1) + \cdots + c(t_{k-1} - t_{k-2}) + c(t_k - t_{k-1}) \\ &\le t_1 + c_0(t_1 - t_0) + c_0 c\,(t_1 - t_0) + \cdots + c^{k-1}(t_1 - t_0) + c^k(t_1 - t_0) \\ &= (1 + c_0 + c_0\,c)\,\eta + (c^3 + c^4 + \cdots + c^k)\,\eta \\ &= \left(1 + (c_0 + 1)\,c + \frac{1 - c^k}{1 - c}\,c^3\right)\,\eta \\ &\le \left(1 + (c_0 + 1)\,c + \frac{c^3}{1 - c}\right)\eta = r_0, \end{aligned} \tag{2.22}$$

which implies $t_{k+1} \le r_0$.

Hence, sequence $\{t_k\}$ is nondecreasing, bounded above, and as such it converges to its unique least upper bound, so that $t^\star \in [0, r_0]$.

We shall show for all $k \ge 0$:

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k, \tag{2.23}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k). \tag{2.24}$$

Let $z_0 \in \overline{U}(x_1, t^\star - t_1)$. Then, we have:

$$\|z - x_0\| \le \|z - x_1\| + \|x_1 - x_0\| \le t^\star - t_1 + t_1 = t^\star - t_0,$$

which implies $z \in \overline{U}(x_0, t^\star - t_0)$. Since also

$$\|x_1 - x_0\| = \|\Gamma\, F(x_0)\| \le \eta = t_1 - t_0,$$

estimate (2.23), and (2.24) hold for $k = 0$.

Given estimates (2.23), (2.24) hold for $n = 1, \ldots, k$, we get:

$$\begin{aligned} \|x_{k+1} - x_0\| &\le \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \\ &\le \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} \le r_0, \end{aligned} \tag{2.25}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \le t_k + \theta(t_{k+1} - t_k) \le t^\star \tag{2.26}$$

for all $\theta \in [0, 1]$.

Using (2.4), (2.14), and (2.25), we obtain:

$$\begin{aligned} \|\Gamma\, [A(x_k) - A(x_0)]\| &\le \omega_0(\|x_k - x_0\|) + \ell_0 \\ &\le \omega_0(t_k) + \ell_0 \\ &\le \omega_0(r_0) + \ell_0 < 1. \end{aligned} \tag{2.27}$$

It follows from (2.27), and the Banach lemma on invertible operators [3,13] that $A(x_k)^{-1}$ exists, and

$$\|A(x_k)^{-1} A(x_0)\| \le (1 - \ell_0 - \omega_0(t_k))^{-1} \le (1 - \ell_0 - \omega_0(r_0))^{-1}. \tag{2.28}$$

In view of (1.2), we obtain the approximation

$$\begin{aligned} x_{k+1} - x_k = -A(x_k)^{-1} A(x_0) \bigg(&\Gamma \int_0^1 (F'(x_k + \theta(x_{k-1} - x_k)) - F'(x_{k-1}))(x_k - x_{k-1}) \mathrm{d}\theta \\ &+ \Gamma\, (F'(x_{k-1}) - A(x_{k-1}))\, (x_k - x_{k-1}) \bigg). \end{aligned} \tag{2.29}$$

By (2.29) for $k = 1$, (2.3), (2.6)–(2.8), (2.18), (2.28), and the induction hypotheses, we obtain:

$$\begin{aligned} \|x_2 - x_1\| &\le \frac{1}{1 - \ell_0 - \omega_0(t_1)} \bigg(\int_0^1 \omega_1((1-\theta)\|x_1 - x_0\|) \|x_1 - x_0\| \mathrm{d}\theta \\ &\quad + (\omega_2(\|x_1 - x_0\|) + \ell_2) \|x_1 - x_0\| \bigg) \\ &\le \frac{H_1\, \omega_1(\eta) + \omega_2(\eta) + \ell_2}{1 - \ell_0 - \omega_0(\eta)}\, \eta = t_2 - t_1. \end{aligned} \tag{2.30}$$

Moreover, using (2.2), (2.3), (2.8), (2.18), (2.28), (2.29), and the induction hypotheses, we get in turn for $k \ge 2$:

$$\begin{aligned} \|x_{k+1} - x_k\| &\le \frac{1}{1 - \ell_0 - \omega_0(t_k)} \bigg(\int_0^1 \omega((1-\theta)\|x_k - x_{k-1}\|) \mathrm{d}\theta \\ &\quad + \omega_2(\|x_k - x_0\|) + \ell_2 \bigg) \|x_k - x_{k-1}\| \\ &\le \frac{H\, \omega(t_k - t_{k-1}) + \omega_2(t_k) + \ell_2}{1 - \ell_0 - \omega_0(t_k)} (t_k - t_{k-1}) = t_{k+1} - t_k, \end{aligned} \tag{2.31}$$

which together with (2.30), shows (2.23) for all $k \ge 0$.

Then, for every $z \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$, we have:

$$\begin{aligned}\|z - x_k\| &\leq \|z - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq t^\star - t_{k+1} + t_{k+1} - t_k = t^\star - t_k,\end{aligned} \tag{2.32}$$

which implies $z \in \overline{U}(x_k, t^\star - t_k)$.

That is (2.24) holds for all $k \geq 0$. The induction for (2.23), and (2.24) is now completed.

In view of (2.23), and (2.24), sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, r_0)$ (since $\overline{U}(x_0, r_0)$ is a closed set).

We shall show $x^\star$ is a solution of Eq. (1.1). We can write:

$$\|\Gamma\, F(x_k)\| \leq \|\Gamma\, A(x_k)\|\, \|A(x_k)^{-1} F(x_k)\|. \tag{2.33}$$

But, we also have:

$$\|A(x_k)^{-1}\, F(x_k)\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{2.34}$$

and

$$\begin{aligned}\|\Gamma\, A(x_k)\| &\leq \|\Gamma\, (A(x_k) - A(x_0))\|\, \|\Gamma\, A(x_0)\| \\ &\leq \omega_0(\|x_k - x_0\|) + 1 \\ &\leq \omega_0(r_0) + 1 = B.\end{aligned} \tag{2.35}$$

In view of (2.33)–(2.35), we get by letting $k \longrightarrow \infty$ that $F(x^\star) = 0$.

Estimate (2.17) follows from (2.16) by using standard majorization techniques [1–3,13].

Finally, to show uniqueness of $x^\star$ in $\mathcal{D}_1$, let us assume $y^\star$ is a solution in $\mathcal{D}_1$.

We need the estimate:

$$\begin{aligned}&\int_0^1 \|\Gamma\, (F'(x^\star + \theta(y^\star - x^\star)) - F'(x_0))\| \leq \int_0^1 (\omega_1(\|x_0 - x^\star - \theta(y^\star - x^\star)\|)\mathrm{d}\theta + \ell_1) \\ &\leq \int_0^1 (\omega_1(\|(1-\theta)(x_0 - x^\star) + \theta(x_0 - y^\star)\|)\mathrm{d}\theta + \ell_1) \\ &\leq \int_0^1 (\omega_1((1-\theta)\|x_0 - x^\star\| + \theta\|x_0 - y^\star\|)\mathrm{d}\theta + \ell_1) \\ &\leq \int_0^{1/2} \omega_1((1-\theta)\|x_0 - x^\star\| + \|x_0 - y^\star\|)\mathrm{d}\theta + \int_{1/2}^1 \omega_1(\theta(\|x_0 - x^\star\| + \|x_0 - y^\star\|))\mathrm{d}\theta + \ell_1 \\ &< \int_0^{1/2} h_1(1-\theta)\omega_1(r_0 + r_1)\mathrm{d}\theta + \int_{1/2}^1 h_1(\theta)\omega_1(r_0 + r_1)\mathrm{d}\theta + \ell_1 \\ &= 2\,\omega_1(r_0 + r_1) \int_{1/2}^1 h_1(\theta)\mathrm{d}\theta + \ell_1 = 1.\end{aligned} \tag{2.36}$$

In view of (2.36), and the Banach lemma on invertible operators,

$$\mathcal{M} = \int_0^1 \Gamma\, F'(x^\star + \theta(y^\star - x^\star))\mathrm{d}\theta$$

is invertible.

We then have:

$$0 = \Gamma\, (F(y^\star) - F(x^\star)) = \Gamma\, \mathcal{M}(y^\star - x^\star),$$

from which it follows

$$x^\star = y^\star.$$

That completes the proof of Theorem 2.1. ◇

Note that if $A(x) = F'(x)$ then in view of (2.5), h_0, ω_0 can replace h_1, ω in the definition of c_0 and t_2, respectively.

3. A extension of Theorem 2.1

In this section, we consider the equation

$$F(x) + G(x) = 0, \tag{3.1}$$

where, F is as in the introduction of this study, and $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a continuous operator.

We use the Newton-like method:

$$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)), \quad (n \geq 0), \tag{3.2}$$

to generate a sequence approximating the solution $x^\star$ of Eq. (3.1).

Then, working along the lines of the proof of Theorem 2.1, we can show the following semilocal result for the Newton-like method (3.2):

Theorem 3.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator, $G : \mathcal{D} \longrightarrow \mathcal{Y}$ a continuous operator, and let $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an approximation of $F'(x)$. Assume that there exist an open convex subset $\mathcal{D}_0$ of $\mathcal{D}$, $x_0 \in \mathcal{D}$, a bounded inverse Γ of A $(= A(x_0))$, continuous, nondecreasing, non-negative functions ω, ω_i, $(i = 1, 2, 3)$ on $[0, +\infty)$, h, h_0 on $[0, 1]$, and non-negative constants η, ℓ_0, ℓ_2, such that, for all $x, y \in \mathcal{D}_0$, $t \in [0, 1]$, $s \in [0, \infty)$, the following conditions hold:* (2.2)–(2.5), (2.8), (2.9), (2.11)–(2.15),

$$\|\Gamma\,(F(x_0) + G(x_0))\| \leq \eta, \tag{3.3}$$

$$\|\Gamma\,[G(x) - G(y)]\| \leq \omega_3(\|x - y\|)\|x - y\|, \tag{3.4}$$

where,

$$c_0 = \frac{H_1\,\omega_1(\eta) + \omega_2(0) + \omega_3(\eta) + \ell_2}{1 - \ell_0 - \omega_0(\eta)}, \qquad c = c(r) = \frac{H\,\omega(\eta) + \omega_2(r) + \omega_3(\eta) + \ell_2}{1 - \ell_0 - \omega_0(r)}.$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by the Newton-like method (3.2) *is well defined, remains in $\overline{U}(x_0, r_0)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, r_0)$ of equation $F(x) + G(x) = 0$.*

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{3.5}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{3.6}$$

where, scalar sequence $\{t_n\}$, $(n \geq 0)$, and $t^\star$ are given by:

$$\begin{aligned} &t_0 = 0, \qquad t_1 = \eta \qquad t_2 = t_1 + c_0\,\eta \\ &t_{n+1} = t_n + \frac{H\,\omega(t_n - t_{n-1}) + \omega_2(t_{n-1}) + \ell_2 + \omega_3(t_n - t_{n-1})}{1 - \ell_0 - \omega_0(t_n)}(t_n - t_{n-1}), \quad (n \geq 2), \end{aligned} \tag{3.7}$$

and

$$t^\star = \lim_{n\to\infty} t_n \leq r_0. \tag{3.8}$$

Furthermore, $x^\star$ is the only solution of (3.1) *in $U(x_0, r_2)$, where r_2 is the unique positive root of equation*

$$f(s) = 0, \tag{3.9}$$

where,

$$f(s) = H\,\omega(s) + \omega_2(s) + \omega_3(s) + \omega_0(s) + \ell_0 + \ell_2 - 1, \tag{3.10}$$

and

$$r_2 \leq r_0. \tag{3.11}$$

Proof. Note that the existence of r_2 is guaranteed by the intermediate value theorem, and (2.9), since $f(0) = \ell_0 + \ell_2 - 1 < 0$, and $f(s) > 0$ for sufficient large $s > 0$. The uniqueness follows from the estimate $f'(s) \geq 0$ for all $s \in [0, \infty)$. That is the graph of function f crosses the positive axis only once.

As is Theorem 2.1, we arrive at (3.6), by simply noticing that there should be extra terms of the form: $\Gamma(Q(x_k) - Q(x_{k-1}))$ (inside the braces in (2.29)), $\omega_3(\eta)$ (at the numerator in (2.30), and (2.31)), since, we are using additional estimate (3.4), and iteration (3.2) instead of (1.2).

Hence, we simply need to show the uniqueness part whose proof differs from the corresponding one in Theorem 2.1.

Let $y^\star$ be a solution in $U(x_0, r_2)$. Using (3.2), we obtain the approximation:

$$\begin{aligned} y^\star - x_{k+1} = {} & A(x_k)^{-1}\, \Gamma^{-1} \Bigg\{ \Gamma \Bigg(\int_0^1 (F'(x_k + \theta(y^\star - x_k)) - F'(x_k))\mathrm{d}\theta \\ & + (F'(x_k) - A(x_k)) \Bigg) (y^\star - x_k) + \Gamma(G(y^\star) - G(x_k)) \Bigg\}. \end{aligned} \tag{3.12}$$

Using (2.2), (2.3), (2.8), (2.28), (3.6), (3.8)–(3.12), we obtain in turn:

$$\begin{aligned} \|y^\star - x_k\| \leq {} & \frac{1}{1 - \ell_0 - \omega_0(\|y^\star - x_k\|)} \Bigg\{ \Bigg(\int_0^1 \|\Gamma(F'(x_k + \theta(y^\star - x_k)) - F'(x_k))\| \mathrm{d}\theta \\ & + \|\Gamma(F'(x_k) - A(x_k))\| \Bigg) \|y^\star - x_k\| + \|\Gamma(G(y^\star) - G(x_k))\| \Bigg\} \\ \leq {} & \frac{1}{1 - \ell_0 - \omega_0(\|y^\star - x_k\|)} \Bigg\{ \int_0^1 \omega(\theta\|y^\star - x_k\|)\mathrm{d}\theta \\ & + \omega_2(\|x_0 - x_k\|) + \ell_2 + \omega_3(\|y^\star - x_k\|) \Bigg\} \|y^\star - x_k\| \\ < {} & \frac{1}{1 - \ell_0 - \omega_0(r_2)} (H\,\omega(r_2) + \omega_2(r_2) + \ell_2 + \omega_3(r_2)) \|x_k - y^\star\| \\ = {} & \|x_k - y^\star\|, \end{aligned} \tag{3.13}$$

which implies $\lim_{k\to\infty} x_k = y^\star$. But we have shown $\lim_{k\to\infty} x_k = x^\star$. Hence, we deduce $x^\star = y^\star$.

That completes the proof of Theorem 3.1. $\diamond$

We can also provide the following local convergence result for the Newton-like method (3.2).

Proposition 3.2. *Assume hypotheses (2.2)–(2.5), (2.8), (2.9), (3.4) hold for x_0 replaced by $x^\star$, and radius of convergence r_2 is given in (3.9).*

Then, sequence $\{x_n\}$ generated by the Newton-like method (3.2) is well defined, remains in $U(x^\star, r_2)$ for all $n \geq 0$, and converges to $x^\star$, provided that $x_0 \in U(x^\star, r_2)$.

Proof. By hypotheses $x_0 \in U(x^\star, r_2)$. Assume $x_n \in U(x^\star, r_2)$ for all $n \leq k$. We shall show $x_{k+1} \in U(x^\star, r_2)$.

As in the proof of Theorem 3.1, using approximation (3.12), we arrive at estimate (3.13) (for $x^\star$ replacing $y^\star$). That is we have:

$$\|x_{k+1} - x^\star\| < \|x_k - x^\star\| \leq r_2,$$

which implies that $x_{k+1} \in U(x^\star, r_2)$, and $\lim_{k\longrightarrow\infty} x_k = x^\star$.

That completes the proof of Proposition 3.2. $\diamond$

Remark 3.3. If $G(x) = 0$ ($x \in \mathcal{D}$), and the function Ω given by (7.6) in [17], is chosen to be equal to ω, then our Proposition 3.2 essentially reduces to Theorem 7.2 in [17].

Note that more general conditions than the ones given in [17], and ours introduced in this study were provided in [2] to show the local (and semilocal) convergence of two-point Newton-like methods (see also [3,4,13]).

4. Special cases and applications

Let us consider the case of Newton's method. That is: $G(x) = 0$, and $A(x) = F'(x)$ for all $x \in \mathcal{D}$.

Then, we have $h_0(s) = h_1(s)$, $\omega_0(s) = \omega_1(s)$, $\omega_2(s) = 0$ for all $s \geq 0$, $\ell_2 = 0$, and $\ell_0 = \ell_1$. We can certainly also set $\ell_0 = \ell_1 = 0$.

In this case, we note:

$$\omega_0(s) \leq \omega(s), \tag{4.1}$$

$$h_0(s) \leq h(s), \tag{4.2}$$

hold for all $s \geq 0$, and $\frac{\omega}{\omega_0}$, $\frac{h}{h_0}$ can be arbitrarily large [1,3,5].

Comparison with a result by Ezquerro, and Hernández [10] (see also [17, Theorem 7.3]): If one reproduces these results in affine invariant form, then the corresponding to $\{t_n\}$ majorizing sequence is essentially given by:

$$v_0 = 0, \qquad v_1 = \eta, \qquad v_{n+1} = v_n + \frac{H\,\omega(v_n - v_{n-1}) + \omega_2(v_{n-1})}{1 - \omega(v_n)}(v_n - v_{n-1}), \quad (n \geq 1). \tag{4.3}$$

Then, if equality holds in (4.1), and (4.2), then our Theorem 3.1 reduces to the corresponding one in [17]. Otherwise it constitutes an improvement under the same computational cost. In particular, we note the following advantages:

(i) Majorizing sequence $\{t_n\}$ is finer than $\{v_n\}$, since an inductive argument shows:

$$t_n < v_n \quad (n \geq 2) \tag{4.4}$$

and

$$t_{n+1} - t_n < v_{n+1} - v_n \quad (n \geq 2). \tag{4.5}$$

(ii) The information on the location of the solution is at least as precise, since:

$$t^\star = \lim_{n\to\infty} t_n \leq \lim_{n\to\infty} v_n = v^\star. \tag{4.6}$$

(iii) The uniqueness radius r_1 is larger than the corresponding one in [17] (since, it is derived from (2.21) for $\omega = \omega_0$, and $h_1 = h$).

It turns out our sufficient convergence conditions are weaker. Indeed, for simplicity, let us consider the case, when

$$\omega(s) = L\,s, \qquad \omega_0(s) = L_0\,s, \quad \text{and} \quad h_0(t) = h(t) = \frac{1}{2}. \tag{4.7}$$

Then, the iterations $\{t_n\}$, $\{v_n\}$ become:

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{n+1} = t_n + \frac{L\,(t_n - t_{n-1})^2}{2\,(1 - L_0\,t_n)}, \quad (n \geq 1), \tag{4.8}$$

$$v_0 = 0, \qquad v_1 = \eta, \qquad v_{n+1} = v_n + \frac{L\,(v_n - v_{n-1})^2}{2\,(1 - L\,v_n)}, \quad (n \geq 1). \tag{4.9}$$

Note that iteration (4.9) converges if the famous for its simplicity and clarity Newton–Kantorovich hypothesis

$$K = L\eta \leq \frac{1}{2} \tag{4.10}$$

holds [13].

However, iteration (4.8) converges under weaker conditions provided that $L_0 < L$.

We need the following result for the convergence of majorizing sequence $\{t_n\}$.

Lemma 4.1. *Assume:*
there exist constants $L_0 \geq 0$, $L \geq 0$ with $L_0 \leq L$, and $\eta \geq 0$, such that:

$$q_0 = \overline{L}\,\eta \leq \frac{1}{2}, \tag{4.11}$$

where,

$$\overline{L} = \frac{1}{8}\left(L + 4\,L_0 + \sqrt{L^2 + 8\,L_0\,L}\right). \tag{4.12}$$

The inequality in (4.11) is strict, if $L_0 = 0$.
Then, sequence $\{t_k\}$ $(k \geq 0)$ given by

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{k+1} = t_k + \frac{L\,(t_k - t_{k-1})^2}{2\,(1 - L_0\,t_k)} \quad (k \geq 1), \tag{4.13}$$

is well defined, nondecreasing, bounded above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star} = \frac{2\,\eta}{2 - \delta}, \tag{4.14}$$

$$1 \leq \delta = \frac{4\,L}{L + \sqrt{L^2 + 8\,L_0\,L}} < 2 \quad \text{for } L_0 \neq 0. \tag{4.15}$$

Moreover, the following estimates hold:

$$L_0\,t^\star \leq 1, \tag{4.16}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \tag{4.17}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\,q_0)^{2^k - 1}\eta, \quad (k \geq 0), \tag{4.18}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\,q_0)^{2^k-1}\,\eta}{1 - (2\,q_0)^{2^k}}, \quad (2\,q_0 < 1),\ (k \geq 0). \tag{4.19}$$

Proof. If $L_0 = 0$, then (4.16) holds trivially. In this case, for $L > 0$, an induction argument shows that

$$t_{k+1} - t_k = \frac{2}{L}(2\,q_0)^{2^k} \quad (k \geq 0).$$

Therefore, we get

$$t_{k+1} = t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) = \frac{2}{L}\sum_{m=0}^{k}(2\,q_0)^{2^m},$$

and

$$t^\star = \lim_{k\to\infty} t_k = \frac{2}{L}\sum_{k=0}^{\infty}(2\,q_0)^{2^k}.$$

Clearly, this series converges, since $k \leq 2^k$, $2\,q_0 < 1$, and is bounded above by the number

$$\frac{2}{L}\sum_{k=0}^{\infty}(2\,q_0)^k = \frac{4}{L\,(2 - L\eta)}.$$

If $L = 0$, then in view of (4.13), $0 \leq L_0 \leq L$, we deduce: $L_0 = 0$, and $t^\star = t_k = \eta$ $(k \geq 1)$.

In the rest of the proof, we assume that $L_0 > 0$.

The result until estimate (4.17) follows from Lemma 1 in [2] (see also [3]).

Note that in particular, Newton–Kantorovich-type convergence condition (4.11) is given in [2] (page 387, Case 3 for δ given by (4.15). The factor η is missing from the left hand side of the inequality three lines before the end of page 387).

In order for us to show (4.18) we need the estimate:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+1}}{1-\frac{\delta}{2}}\eta \le \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right) \quad (k \ge 1). \tag{4.20}$$

For $k = 1$, estimate (4.20) becomes

$$\left(1+\frac{\delta}{2}\right)\eta \le \frac{4\overline{L}-L}{4\,\overline{L}L_0},$$

or

$$\left(1+\frac{2\,L}{L+\sqrt{L^2+8\,L_0\,L}}\right)\eta \le \frac{4\,L_0-L+\sqrt{L^2+8\,L_0\,L}}{L_0(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})}.$$

In view of hypothesis (4.11), it suffices to show:

$$\frac{L_0(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})(3\,L+\sqrt{L^2+8\,L_0\,L})}{(L+\sqrt{L^2+8\,L_0\,L})(4\,L_0-L+\sqrt{L^2+8\,L_0\,L})} \le 2\overline{L},$$

which is true as equality.

Let us now assume estimate (4.20) is true for all integers smaller or equal to k. We must show (4.20) holds for k replaced by $k+1$:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+2}}{1-\frac{\delta}{2}}\eta \le \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right) \quad (k \ge 1),$$

or

$$\left(1+\frac{\delta}{2}+\left(\frac{\delta}{2}\right)^2+\cdots+\left(\frac{\delta}{2}\right)^{k+1}\right)\eta \le \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right). \tag{4.21}$$

By the induction hypothesis to show estimate (4.21), it suffices to have:

$$\frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right)+\left(\frac{\delta}{2}\right)^{k+1}\eta \le \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right),$$

or

$$\left(\frac{\delta}{2}\right)^{k+1}\eta \le \frac{1}{L_0}\left(\left(\frac{\delta}{2}\right)^{k-1}-\left(\frac{\delta}{2}\right)^{k}\right)\frac{L}{4\,\overline{L}},$$

or

$$\delta^2\eta \le \frac{L(2-\delta)}{2\,\overline{L}\,L_0}.$$

In view of hypothesis (4.11), we can show instead:

$$\frac{2\,\overline{L}\,L_0\,\delta^2}{L(2-\delta)} \le 2\overline{L},$$

which holds as equality by the choice of δ given in (4.15).

That completes the induction for estimate (4.20).

We shall show (4.18) using induction on $k \geq 0$: Estimate (4.18) is true for $k = 0$ by (4.11), (4.13), and (4.15). In order for us to show estimate (4.18) for $k = 1$, since $t_2 - t_1 = \frac{L(t_1-t_0)^2}{2(1-L_0 t_1)}$, it suffices:

$$\frac{L\eta^2}{2(1-L_0\eta)} \leq \delta \bar{L} \eta^2,$$

or

$$\frac{L}{1-L_0\eta} \leq \frac{8\,\bar{L}\,L}{L+\sqrt{L^2+8\,L_0\,L}} \quad (\eta \neq 0),$$

or

$$\eta \leq \frac{1}{L_0}\left(1 - \frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\bar{L}}\right) \quad (L_0 \neq 0, L \neq 0).$$

But by (4.11) we have:

$$\eta \leq \frac{4}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}}.$$

It then suffices to show

$$\frac{4}{L+4L_0+\sqrt{L^2+8\,L_0\,L}} \leq \frac{1}{L_0}\left(1 - \frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\bar{L}}\right),$$

or

$$\frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\bar{L}} \leq 1 - \frac{4\,L_0}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}},$$

or

$$\frac{L+\sqrt{L^2+8L_0L}}{8\bar{L}} \leq \frac{L+\sqrt{L^2+8L_0L}}{L+4L_0+\sqrt{L^2+8L_0L}},$$

which is true as equality by (4.12).

Let us assume (4.21) holds for all integers smaller or equal to k. We shall show (4.21) holds for k replaced by $k+1$.

Using (4.13), and the induction hypothesis, we have in turn

$$\begin{aligned} t_{k+2} - t_{k+1} &= \frac{L}{2(1-L_0\,t_{k+1})}(t_{k+1}-t_k)^2 \\ &\leq \frac{L}{2(1-L_0\,t_{k+1})}\left(\left(\frac{\delta}{2}\right)^k (2q_0)^{2^k-1}\eta\right)^2 \\ &\leq \frac{L}{2(1-L_0\,t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\,q_0)^{-1}\eta\right)\left(\left(\frac{\delta}{2}\right)^{k+1} (2\,q_0)^{2^{k+1}-1}\eta\right) \\ &\leq \left(\frac{\delta}{2}\right)^{k+1} (2\,q_0)^{2^{k+1}-1}\eta, \end{aligned}$$

since,

$$\frac{L}{2(1-L_0\,t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\,q_0)^{-1}\eta\right) \leq 1, \quad (k \geq 1). \tag{4.22}$$

Indeed, we can show instead of (4.22):

$$t_{k+1} \leq \frac{1}{L_0}\left(1 - \left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\bar{L}}\right),$$

which is true, since by (4.17), and the induction hypotheses:

$$\begin{aligned} t_{k+1} &\leq t_k + \frac{\delta}{2}(t_k - t_{k-1}) \\ &\leq t_1 + \frac{\delta}{2}(t_1 - t_0) + \cdots + \frac{\delta}{2}(t_k - t_{k-1}) \\ &\leq \eta + \left(\frac{\delta}{2}\right)\eta + \cdots + \left(\frac{\delta}{2}\right)^k \eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{k+1}}{1 - \frac{\delta}{2}}\eta \\ &\leq \frac{1}{L_0}\left(1 - \left(\frac{\delta}{2}\right)^{k-1} \frac{L}{4\overline{L}}\right). \end{aligned}$$

That completes the induction for estimate (4.18).

Using estimate (4.21) for $j \geq k$, we obtain in turn for $2\,q_0 < 1$:

$$\begin{aligned} t_{j+1} - t_k &= (t_{j+1} - t_j) + (t_j - t_{j-1}) + \cdots + (t_{k+1} - t_k) \\ &\leq \left(\left(\frac{\delta}{2}\right)^j (2\,q_0)^{2^j-1} + \left(\frac{\delta}{2}\right)^{j-1} (2\,q_0)^{2^{j-1}-1} + \cdots + \left(\frac{\delta}{2}\right)^k (2\,q_0)^{2^k-1}\right)\eta \\ &\leq \left(1 + (2\,q_0)^{2^k} + \left((2\,q_0)^{2^k}\right)^2 + \cdots\right)\left(\frac{\delta}{2}\right)^k (2\,q_0)^{2^k-1}\eta \\ &= \left(\frac{\delta}{2}\right)^k \frac{(2\,q_0)^{2^k-1}\eta}{1 - (2\,q_0)^{2^k}}. \end{aligned} \tag{4.23}$$

Estimate (4.19) follows from (4.23) by letting $j \longrightarrow \infty$.

That completes the proof of Lemma 4.1. $\diamond$

Remark 4.2. If $L_0 = L$, Lemma 4.1 provides the usual error bounds appearing essentially in the Newton–Kantorovich theorem [13].

However, if $L_0 < L$, then our sufficient convergence condition (4.11) is weaker than (4.10). Finally, our ratio $2\,q_0$ is also smaller than $2K$.

Let us provide two examples where (4.10) is violated, but (4.11) holds.

Example 4.3. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, equipped with the max-norm, and

$$x_0 = (1, 1)^T, \qquad \mathcal{D}_0 = \{x : \|x - x_0\| \leq 1 - \beta\}, \quad \beta \in \left[0, \frac{1}{2}\right).$$

Define function F on $\mathcal{D}_0$ by

$$F(x) = (\xi_1^3 - \beta, \xi_2^3 - \beta)^T, \qquad x = (\xi_1, \xi_2)^T. \tag{4.24}$$

The Fréchet-derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\xi_1^2 & 0 \\ 0 & 3\xi_2^2 \end{bmatrix}.$$

Using hypotheses of Theorem 2.1, we get:

$$\eta = \frac{1}{3}(1 - \beta), \qquad L_0 = 3 - \beta, \quad \text{and} \quad L = 2\,(2 - \beta).$$

The Newton–Kantorovich condition (4.6) is violated, since

$$\frac{4}{3}(1-\beta)(2-\beta) > 1 \quad \text{for all } \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that (NTM) converges to $x^\star = (\sqrt[3]{\beta}, \sqrt[3]{\beta})^T$, starting at x_0.

However, our condition (4.11) is true for all $\beta \in I = \left[.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 2.1 can apply to solve Eq. (4.24) for all $\beta \in I$.

Example 4.4. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ be equipped with the ℓ_∞-norm [14, p. 41]. Choose $x_0 = [1, 1]^T$, $\mathcal{D}_0 = \{x : \|x - x_0\| \leq 1 - b\}$ for $b \in [0, 1)$, and define function $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ on $\mathcal{D}_0$, where,

$$\begin{aligned} F_1(v, w) &= v^3 + \epsilon_1 w - b \\ F_2(v, w) &= w^3 + \epsilon_2 v - b \end{aligned} \tag{4.25}$$

for some given constants ϵ_1 and ϵ_2, such that $\epsilon_1\epsilon_2 \neq 9$.

Then, the Fréchet-derivative F' of operator F is given by

$$F'(v, w) = \begin{bmatrix} 3v^2 & \epsilon_1 \\ \epsilon_2 & 3w^2 \end{bmatrix},$$

and

$$F'(v, w) - F'(\overline{v}, \overline{w}) = \begin{bmatrix} 3(v^2 - \overline{v}^2) & 0 \\ 0 & 3(w^2 - \overline{w}^2) \end{bmatrix}.$$

If $9v^2w^2 \neq \epsilon_1\epsilon_2$, then, we obtain

$$F'(v, w)^{-1} = \frac{1}{9v^2w^2 - \epsilon_1\epsilon_2} \begin{bmatrix} 3w^2 & -\epsilon_1 \\ -\epsilon_2 & 3v^2 \end{bmatrix},$$

and, in particular for x_0:

$$F'(v_0, w_0)^{-1} = \frac{1}{9 - \epsilon_1\epsilon_2} \begin{bmatrix} 3 & -\epsilon_1 \\ -\epsilon_2 & 3 \end{bmatrix}. \tag{4.26}$$

We need the estimate

$$\begin{aligned} &\frac{1}{9 - \epsilon_1\epsilon_2} \begin{bmatrix} 3 & -\epsilon_1 \\ -\epsilon_2 & 3 \end{bmatrix} \begin{bmatrix} 3(v^2 - \overline{v}^2) & 0 \\ 0 & 3(w^2 - \overline{w}^2) \end{bmatrix} \\ &\quad = \frac{1}{9 - \epsilon_1\epsilon_2} \begin{bmatrix} 3(v^2 - \overline{v}^2) & -\epsilon_1(w^2 - \overline{w}^2) \\ -\epsilon_2(v^2 - \overline{v}^2) & 3(w^2 - \overline{w}^2) \end{bmatrix}. \end{aligned} \tag{4.27}$$

Set

$$\epsilon = \max\{3 + |\epsilon_1|, 3 + |\epsilon_2|\}. \tag{4.28}$$

Using (4.26)–(4.28), we obtain the Lipschitz constants:

$$L = \frac{6\epsilon(2-b)}{|9 - \epsilon_1\epsilon_2|}, \tag{4.29}$$

and

$$L_0 = \frac{3\epsilon(3-b)}{|9 - \epsilon_1\epsilon_2|}. \tag{4.30}$$

Moreover, by (4.25), and (4.26), we can set

$$\eta = \frac{1}{|9 - \epsilon_1\epsilon_2|} \max\{\overline{\alpha}, \overline{\beta}\}, \tag{4.31}$$

where,

$$\overline{\alpha} = |3 + 2\,\epsilon_1 - 3\,b - \epsilon_1\,\epsilon_2 + \epsilon_1\,b|, \tag{4.32}$$

$$\overline{\beta} = |3 + 2\,\epsilon_2 - 3\,b - \epsilon_1\epsilon_2 + \epsilon_2\,b|. \tag{4.33}$$

Let us choose for example

$$b = .49, \qquad \epsilon_1 = .01, \quad \text{and} \quad \epsilon_2 = .02. \tag{4.34}$$

Using (4.28)–(4.34), we obtain

$$L = 3.040200893, \qquad L_0 = 2.526789485, \qquad \eta = .175515011, \qquad \delta = 1.06222409,$$
$$\overline{\alpha} = 1.5547, \qquad \overline{\beta} = 1.5796, \qquad \overline{L} = 2.69449131, \quad \text{and} \quad t^{\star\star} = .374321859.$$

Condition (4.10) is violated, since

$$K = .533600893 > \frac{1}{2}.$$

Hence, there is no guarantee that Newton's method starting at x_0 converges to a zero $x^\star$ of function F.

However, our condition (4.11) is satisfied, since

$$q_0 = .472916269 < \frac{1}{2}.$$

Moreover, we have:

$$\overline{U}(x_0, t^{\star\star}) \subseteq \mathcal{D}_0.$$

In view of all the above the hypotheses of Theorem 2.1 are satisfied.

That is our Theorem 2.1 guarantees the existence of a zero $x^\star$ in $\overline{U}(x_0, t^{\star\star})$ of function F, which can be obtained as the limit of sequence $\{x_n\}$.

We finally provide two more examples, where $L_0 < L$.

Example 4.5. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with norm

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t)\, u(t) \mathrm{d}t + y(s) - \theta. \tag{4.35}$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. Equations of the form (4.35) arise in the kinetic theory of gasses [3]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s, t) = \frac{s}{s+t}$, for all $s \in [0, 1]$, and $t \in [0, 1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s, t)\, x(t) \mathrm{d}t + y(s) - \theta, \tag{4.36}$$

for all $s \in [0, 1]$, then every zero of F satisfies Eq. (4.35).

We have the estimates:

$$\max_{0\le s\le 1} \left| \int \frac{s}{s+t} \mathrm{d}t \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from hypotheses of Theorem 2.1 that

$$\eta = \xi(|\lambda|\ \ln 2 + 1 - \theta),$$
$$L = 2\xi(|\lambda|\ \ln 2 + 3(2-\theta)) \quad \text{and} \quad L_0 = \xi(2|\lambda|\ \ln 2 + 3(3-\theta)).$$

It follows from Theorem 2.1 that if condition (4.11) holds, then problem (4.35) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (4.10).

Note also that $L_0 < L$ for all $\theta \in [0, 1]$.

Example 4.6. Consider the following nonlinear boundary value problem [3]

$$\begin{cases} u'' = -u^3 - \gamma u^2 \\ u(0) = 0, \qquad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)(u^3(t) + \gamma u^2(t))\mathrm{d}t \tag{4.37}$$

where, Q is Green's function:

$$Q(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\le s\le 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, with norm

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Then problem (4.37) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\mathrm{d}t,$$

and

$$G(x)(s) = 0.$$

It is easy to verify that the Fréchet-derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\mathrm{d}t.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\gamma < 5$, then

$$\|I - F'(u_0)\| \le \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3+2\gamma}{8},$$
$$\|F'(u_0)^{-1}\| \le \frac{1}{1 - \frac{3+2\gamma}{8}} = \frac{8}{5-2\gamma},$$
$$\|F(u_0)\| \le \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1+\gamma}{8},$$
$$\|F(u_0)^{-1}F(u_0)\| \le \frac{1+\gamma}{5-2\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\mathrm{d}t.$$

Consequently,

$$\begin{aligned}
\|F'(x) - F'(y)\| &\le \frac{\|x-y\|(2\gamma + 3(\|x\| + \|y\|))}{8} \\
&\le \frac{\|x-y\|(2\gamma + 6R + 6\|u_0\|)}{8} \\
&= \frac{\gamma + 6R + 3}{4}\|x-y\|, \\
\|F'(x) - F'(u_0)\| &\le \frac{\|x-u_0\|(2\gamma + 3(\|x\| + \|u_0\|))}{8} \\
&\le \frac{\|x-u_0\|(2\gamma + 3R + 6\|u_0\|)}{8} \\
&= \frac{2\gamma + 3R + 6}{8}\|x-u_0\|.
\end{aligned}$$

Therefore, conditions of Theorem 2.1 hold with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \qquad L = \frac{\gamma + 6R + 3}{4}, \qquad L_0 = \frac{2\gamma + 3R + 6}{8}.$$

Note also that $L_0 < L$.

5. Conclusion

We provided a semilocal convergence analysis for Newton-like methods using ω-type conditions, in order to approximate a locally unique solution of an equation in a Banach space.

Using a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions, we provided an analysis with the following advantages over the works in [10,17]: weaker sufficient convergence conditions, tighter error bounds and larger convergence domain in some interesting cases. Numerical examples and applications further validating the results are also provided in this study.

References

[1] I.K. Argyros, On the Newton–Kantorovich hypothesis for solving equations, J. Comput. Appl. Math. 169 (2004) 315–332.
[2] I.K. Argyros, A unifying local–semilocal convergence analysis and applications for two-point Newton-like methods in Banach space, J. Math. Anal. Appl. 298 (2004) 374–397.
[3] I.K. Argyros, Computational Theory of Iterative Methods, in: C.K. Chui, L. Wuytack (Eds.), Series: Studies in Computational Mathematics, vol. 15, Elsevier Publ. Co., New York, USA, 2007.
[4] I.K. Argyros, S. Hilout, Efficient methods for solving equations and variational inequalities, Polimetrica Publisher, Milano, Italy, 2009.
[5] I.K. Argyros, S. Hilout, Aspects of the computational theory for certain iterative methods, Polimetrica Publisher, 2009.
[6] I.K. Argyros, S. Hilout, On the weakening of the convergence of Newton's method using recurrent functions, J. Complexity 25 (2009) 530–543.
[7] X. Chen, T. Yamamoto, Convergence domains of certain iterative methods for solving nonlinear equations, Numer. Funct. Anal. Optim. 10 (1989) 37–48.
[8] J.E. Dennis, Toward a unified convergence theory for Newton-like methods, in: L.B. Rall (Ed.), Nonlinear Functional Analysis and Applications, Academic Press, New York, 1971, pp. 425–472.
[9] P. Deuflhard, G. Heindl, Affine invariant convergence theorems for Newton's method and extensions to related methods, SIAM J. Numer. Anal. 16 (1979) 1–10.
[10] J.A. Ezquerro, M.A. Hernández, Generalized differentiability conditions for Newton's method, IMA J. Numer. Anal. 22 (2002) 187–205.
[11] J.M. Gutiérrez, A new semilocal convergence theorem for Newton's method, J. Comput. Appl. Math. 79 (1997) 131–145.
[12] Z. Huang, A note of Kantorovich theorem for Newton iteration, J. Comput. Appl. Math. 47 (1993) 211–217.
[13] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.

[14] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
[15] F.A. Potra, Sharp error bounds for a class of Newton-like methods, Libertas Math. 5 (1985) 71–84.
[16] P.D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newton's method, J. Complexity 25 (2009) 38–62.
[17] P.D. Proinov, New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems, J. Complexity 26 (2010) 3–42.
[18] W.C. Rheinboldt, A unified convergence theory for a class of iterative processes, SIAM J. Numer. Anal. 5 (1968) 42–63.
[19] T. Yamamoto, A convergence theorem for Newton-like methods in Banach spaces, Numer. Math. 51 (1987) 545–557.
[20] P.P. Zabrejko, D.F. Nguen, The majorant method in the theory of Newton–Kantorovich approximations and the Pták error estimates, Numer. Funct. Anal. Optim. 9 (1987) 671–684.